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ANTIPLANE DEFORMATION OF A PLANE WITH A SLIT FILLED WITH A MATERIAL WITH A LOW SHEAR MODULUS

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1. We will assume that the slit occupies the section $(-a, a)$ of the x axis. The mathematical problem is formulated as follows (we will denote the upper half-plane ($y > 0$) by R_+^2 and the lower half-plane ($y < 0$) by R_-^2): it is required to determine the harmonic function u_1 and u_2 in the upper half-plane and the lower half-plane, respectively, such that

$$u_1 = u_2,\ u_{1y}(x, +0) = u_{2y}(x, -0) \text{ for } |x| \geqslant a; \tag{1.1}$$

$$u_{1y}(x, +0) = k[u] + f(x),\ u_{2y}(x, -0) = k[u],\ |x| \leqslant a, \tag{1.2}$$

where $[u] = u_1(x, +0) - u_2(x, -0)$; $k > 0$. It is assumed the stresses vanish at infinity.

Conditions (1.1) are for ideal contact when $|x| \geq a$, and conditions (1.2) arise if we assume that there is a thin layer of material with a low shear modulus on the section of the interface $(-a, a)$ (see the derivation of this conjugate condition in [1, 2]). The contact between a body and a thin poorly conducting layer corresponds to this problem in the case of steady heat conduction [2].

In this paper we construct a system of hypersingular equations for problem (1.1), (1.2) and we investigate its behavior for small and large k. It is shown that a solution of this system of equations exists in a certain Hölder-continuous weighted class of functions. In the theory of elasticity the conditions of viscous friction at the interface correspond to conditions (1.2) and k is called the coefficient of viscosity. We will adhere to this terminology below.

The function f(x) in (1.2) models the action of specified surface forces.

2. We will now give a formal derivation of the system of integral equations for problem (1.1), (1.2). In view of the fact that the stresses vanish at infinity it is natural to seek u_1 and u_2 as a Poisson integral for the lower and upper half-planes respectively

$$u_1 = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{m_1(t)y\,dt}{(x-t)^2 + y^2}; \tag{2.1}$$

$$u_2 = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{m_2(t)y\,dt}{(x-t)^2 + y^2} \tag{2.2}$$

with as yet unknown densities m_1 and m_2. It follows from (2.1) and (2.2) that $u_1(x, +0) = m_1(x)$, $u_2(x, -0) = m_2(x)$, and therefore when $|x| \geq a$, $m_1(x) = m_2(x)$. By calculating $u_{1y}(x, +0)$ and $u_{2y}(x, -0)$ we obtain

$$\int_{-\infty}^{\infty} \frac{m_1(t) + m_2(t)dt}{(x-t)^2} = 0$$

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when $|x| \geq a$, but then it is natural to assume that $m_1(x) + m_2(x) = 0$ and $m_1(x) = m_2(x) = 0$ when $|x| \geq a$. The interval of integration $(-\infty, \infty)$ in this case contracts to the interval $(-a, a)$ and the conjugation conditions (1.2) lead to the following system of equations:

$$-\frac{1}{\pi}\int_{-a}^{+a}\frac{m_2(t)dt}{(x-t)^2} = k(m_1(x) - m_2(x)); \tag{2.3}$$

$$+\frac{1}{\pi}\int_{-a}^{+a}\frac{m_1(t)dt}{(x-t)^2} = k(m_1(x) - m_2(x)) + f(x) . \tag{2.4}$$

Note that

$$Dm(x) = \frac{1}{\pi}\int_{-a}^{+a}\frac{m(t)dt}{(x-t)^2}$$

is a hypersingular integral and it must be understood as the finite part of the diverging Hadamard integral [3]. When $f \in C^{1,\alpha}(a,$ b) is a Banach space of functions that are continuous and have a first continuous derivative in the section $(a,$ b), where the first derivative satisfies the Hölder condition with index $\alpha <$ Dm(x) can be calculated from the from formula

$$\text{p.f.}\int_a^b\frac{f(t)dt}{(x-t)^2} = -\frac{f(a)}{x-a} - \frac{f(b)}{b-x} - \text{v.p.}\int_a^b\frac{f'(t)dt}{x-t} .$$

Here p.f. and v.p. are the finite part of the Hadamard diverging integral and the principal value of the Cauchy singular integral, respectively. We will put $m(x) = m_1(x) + m_2(x)$, $s(x) = m_1(x)$. After adding and subtracting equations (2.3) and (2.4) we obtain the two equivalent equations

$$\frac{1}{\pi}\int_{-a}^{+a}\frac{s(t)dt}{(x-t)^2} = 2ks(x) + f(x); \tag{2.5}$$

$$\frac{1}{\pi}\int_{-a}^{+a}\frac{m(t)dt}{(x-t)^2} = f(x) . \tag{2.6}$$

We introduce the Banach space $D^{1,\alpha}$ of the functions s(x) which vanish when $x = \pm a$, such that $s'(x) = g'^*(x)(a^2 - x^2)^{-1/2}(g'^*(x) \in {}^{1,\alpha})$ with norm

$$\|\varphi\|_{D^{1,\alpha}} = \|\varphi\|_C + \|\varphi'^*\|_{C^{0\alpha}}.$$

Consider Eq. (2.5). It can be written as

$$\frac{d}{dx}\int_{-a}^{+a}\frac{s(t)dt}{t-x} = 2ks(x) + f(x) . \tag{2.7}$$

By considering the singular integral in (2.7) in the class of functions that are bounded when $x = \pm a$, we obtain

$$s(x) = -\frac{1}{\pi}\sqrt{a^2-x^2}\int_{+a}^{-a}\frac{(2k\int_{-a}^{t} s(\alpha)d\alpha + F(x) + C)dt}{\sqrt{a^2-t^2}(t-x)} \tag{2.8}$$

when the following condition of solvability is satisfied:

$$\int_{+a}^{-a} \frac{(2k \int_{-a}^{t} s(\alpha)d\alpha + F(x) + C)dt}{\sqrt{a^2 - t^2}} = 0,$$

which enables us to determine the constant C. By changing the order of integration in (2.8), we have the following Fredholm integral equation of the second kind for s(x):

$$s(x) = -\frac{k}{\pi}\int_{-a}^{+a} s(t)k(t,x)dt - \frac{1}{\pi}\int_{-a}^{+a} f(t)k(t,x)dt.$$

Here

$$k(t,x) = \frac{1}{2}\ln\frac{a^2 - tx + \sqrt{(a^2 - x^2)(a^2 - t^2)}}{a^2 - tx - \sqrt{(a^2 - x^2)(a^2 - t^2)}};$$

the kernel $k(x, t) \in L^2(-a, a)$. Note that (2.5) can be reduced to the Prandtl equation in thin-wing theory by putting $s(x) = g'(x)$.

Equation (2.6) has an explicit solution that is bounded when $x = \pm a$. In fact, by integrating (2.6) with respect to x we obtain

$$\frac{1}{\pi}\int_{-a}^{+a}\frac{m(t)dt}{t - x} = \int_{-a}^{x} f(t)dt + C = F(x) + C,$$

$$m(x) = -\frac{1}{\pi}\sqrt{a^2 - x^2}\int_{-a}^{+a}\frac{F(t)dt}{\sqrt{a^2 - t^2}(t - x)},$$

if the following condition of solvability in the class of bounded functions is satisfied:

$$C = -\frac{1}{\pi}\int_{-a}^{+a}\frac{F(x)dx}{\sqrt{a^2 - x^2}}.$$

The results obtained in [4] show that the solutions of Eqs. (2.6) belong to $D^{1,\alpha}$ when $f(x) \in C^{0,\alpha}$. We recall that in this case m_1 and m_2 vanish at the ends of the interval, and their derivatives may have root singularities. As shown in [4], the solution of Eq. (2.5) for small k can be constructed either by the method of successive approximations or in the form of a converging power series in k. Moreover, a method is also given there for reducing Prandtl's equation to an equivalent infinite system of linear algebraic equations which is also applicable in the case being investigated here. For large values of k, Eq. (2.5) becomes a singularly perturbed equation and its approximate solution can be obtained by the method of joined asymptotic expansions. Put $k = \varepsilon^{-1}$. Equation (2.5) can then be written in the form

$$\frac{\varepsilon}{\pi}\int_{-a}^{+a}\frac{s(t)dt}{(x - t)^2} = 2s(x) + \varepsilon f(x). \tag{2.9}$$

By representing the solution of this equation as a series in powers of

$$s(x) = \sum_{n=0}^{\infty}\varepsilon^n s_n(x),$$

we obtain that $s_0(x) = 0$, and

$$s_1(x) = -\frac{1}{2}f(x), \; s_n(x) = \frac{1}{2\pi}\int_{-a}^{+a}\frac{s_{n-1}(t)dt}{(x - t)^2}, \; n > 1,$$

and hence it is impossible to satisfy the boundary conditions that make s(x) vanish at $x = \pm a$. A method of constructing the asymptotic form of the solution of Eq. (2.9) for small ε is described in detail in [5].

Note that in this problem, unlike the usual problems in the theory of cracks, only the shear stress τ, has a root singularity when $x = \pm a$.

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